

**THE HOMOTOPY TYPE OF THE SPACE OF DEGREE 0
IMMERSED PLANE CURVES**

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ABSTRACT. The space $B_i^0 = \text{Imm}^0(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$ of all immersions of rotation degree 0 in the plane modulo reparametrizations has homotopy groups $\pi_1(B_i^0) = \mathbb{Z}$, $\pi_2(B_i^0) = \mathbb{Z}$, and $\pi_k(B_i^0) = 0$ for $k \geq 3$.

1. INTRODUCTION

For an immersion $c : S^1 \rightarrow \mathbb{R}^2$ the (rotation) degree is the winding number of $c' : S^1 \rightarrow \mathbb{R}^2$ around 0. Let $\text{Imm}^k(S^1, \mathbb{R}^2)$ denote the connected smooth Fréchet manifold of all immersions of degree k . It was shown in [4] that for $k \neq 0$ the space $\text{Imm}^k(S^1, \mathbb{R}^2)$ contains a copy of S^1 as a smooth strong deformation retract and that the infinite dimensional orbifold $\text{Imm}^k(S^1, \mathbb{R}^2)/\text{Diff}^+(S^1)$ is contractible, where $\text{Diff}^+(S^1)$ denotes the regular Fréchet Lie group of orientation preserving diffeomorphisms. The proof in [4] consists in expanding the classical proof of the theorem of Whitney and Graustein (see [1], [2], [8]) into the construction of an S^1 -equivariant smooth deformation retraction. For $k = 0$ this did not work.

In this paper we treat the case $k = 0$. In section 2 we first give simple argument which shows that $\pi_1(B^0(S^1, \mathbb{R}^2))$ contains \mathbb{Z} . In section 3 we give a more involved proof that $\text{Imm}^0(S^1, \mathbb{R}^2)$ is homotopy equivalent to S^1 . In section 4 we show that factoring out $\text{Diff}^+(S^1)$ gives a fibration with homotopically trivially embedded fiber, and then the homotopy sequence shows that $\pi_1(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}$, $\pi_2(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}$, and $\pi_k(B^{0,+}(S^1, \mathbb{R}^2)) = 0$ for $k > 2$. Factoring out the larger group $\text{Diff}(S^1)$ gives a two-sheeted covering and the final result.

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2. A SIMPLE PROOF THAT $\mathbb{Z} \subseteq \pi_1(B^0(S^1, \mathbb{R}^2))$

2.1. Proposition. $\text{Imm}^0(S^1, \mathbb{R}^2)/\text{Diff}^+(S^1)$ is not contractible.

Proof. We shall view a curve $c \in \text{Imm}(S^1, \mathbb{R}^2)$ as a 2π -periodic plane valued function. A smooth function $a = a(c, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is called an argument of a curve c if

$$\frac{c'(\theta)}{|c'(\theta)|} = \exp(ia(\theta));$$

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it is unique up to addition of an inter multiple of 2π . If the curve c has degree k then $a(\theta + 2\pi) - a(\theta) = 2k\pi$. Thus, a curve c is in $\text{Imm}^0(S^1, \mathbb{R}^2)$ if and only if some (any) argument of c is 2π -periodic. For a curve $c \in \text{Imm}^0(S^1, \mathbb{R}^2)$, we define the *average argument* $\alpha(c) \in S^1$ by

$$\alpha(c) = \exp\left(\frac{i}{l(c)} \int_0^{2\pi} a(c, \theta) |c'(\theta)| d\theta\right),$$

which does not depend on the choice of $a(c, \cdot)$ and defines a well-defined smooth mapping $\alpha : \text{Imm}^0(S^1, \mathbb{R}^2) \rightarrow S^1$. Also, since any argument a of a degree 0 curve is 2π -periodic, $\alpha(c)$ is invariant under the action of $\text{Diff}^+(S^1)$. So we can view α as a map

$$\alpha : B^{0,+}(S^1, \mathbb{R}^2) = \text{Imm}^0(S^1, \mathbb{R}^2) / \text{Diff}^+(S^1) \rightarrow S^1.$$

For $\varphi \in S^1 \subset \mathbb{C} = \mathbb{R}^2$, the rotation map $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ act on $B^{0,+}(S^1, \mathbb{R}^2)$ and obviously

$$\alpha(\varphi.c) = \varphi.\alpha(c).$$

So choosing a free orbit $S^1.C$ for the rotation action of S^1 on $B^{0,+}(S^1, \mathbb{R}^2)$, the composition

$$S^1.C \hookrightarrow \text{Imm}^0(S^1, \mathbb{R}^2) / \text{Diff}^+(S^1) \xrightarrow{\alpha} S^1$$

equals the identity on S^1 , thus $\pi_1(S^1) = \mathbb{Z} \subset \pi_1(B^{0,+}(S^1, \mathbb{R}^2))$.

Moreover, $\alpha(c(-)) = -\alpha(c)$ implies that α factors as follows, where the vertical arrows are 2-sheeted coverings:

$$\begin{array}{ccc} B^{0,+}(S^1, \mathbb{R}^2) & \xlongequal{\quad} & \text{Imm}^0(S^1, \mathbb{R}^2) / \text{Diff}^+(S^1) \xrightarrow{\alpha} S^1 \\ & & \downarrow 2 \qquad \qquad \qquad \downarrow 2 \\ B^0(S^1, \mathbb{R}^2) & \xlongequal{\quad} & \text{Imm}^0(S^1, \mathbb{R}^2) / \text{Diff}(S^1) \xrightarrow{\bar{\alpha}} S^1 \end{array}$$

Thus we also get in a similar way $\pi_0(S^1) = \mathbb{Z} \subset \pi_0(B^0(S^1, \mathbb{R}^2))$. \square

3. THE HOMOTOPY TYPE OF $\text{Imm}^0(S^1, \mathbb{R}^2)$

3.1. Proposition. *The space $\text{Imm}^0(S^1, \mathbb{R}^2)$ of degree 0 immersions in the plane is homotopy equivalent to S^1 .*

Proof. This will follow from **3.2** – **3.5** below. \square

3.2. Let $\text{Imm}^{0,*}(S^1, \mathbb{R}^2) := \{c \in \text{Imm}^0(S^1, \mathbb{R}^2); c(0) = 0\}$. Clearly we have $\text{Imm}^0(S^1, \mathbb{R}^2) \cong \text{Imm}^{0,*}(S^1, \mathbb{R}^2) \times \mathbb{R}^2$ and $\text{Imm}^0(S^1, \mathbb{R}^2) \sim \text{Imm}^{0,*}(S^1, \mathbb{R}^2)$, where \cong denotes homeomorphism and \sim homotopy equivalent. Let us define a map

$$\begin{aligned} \Phi : \text{Imm}^{0,*} &\rightarrow C^\infty(S^1, \mathbb{R}_+) \times C^\infty(S^1, S^1) \\ \Phi(c)(\theta) &= \left(|c_\theta(\theta)|, \frac{c_\theta(\theta)}{|c_\theta(\theta)|} \right) =: (v(\theta), e(\theta)). \end{aligned}$$

The map Φ is injective. For $(v, e) = \Phi(c)$, the winding number of e equals the degree 0 of c and thus $\int_0^{2\pi} v.e d\theta = 0$.

Lemma. *The length of the image of e is greater than π .*

Proof. If not, there exists a number $r \in \mathbb{R}$ such that

$$\exp(ir) \in \text{Im}(e) \subset \exp(i[r - \pi/2, r + \pi/2]).$$

Then, $\langle \exp(ir), e(\theta) \rangle$ is nonnegative for any θ and strictly positive for some θ . Therefore $\int_0^{2\pi} \langle \exp(ir), v.e \rangle d\theta > 0$. This contradicts

$$\int_0^{2\pi} \langle \exp(ir), v.e \rangle d\theta = \left\langle \exp(ir), \int_0^{2\pi} v.e d\theta \right\rangle = \langle \exp(ir), 0 \rangle = 0. \quad \square$$

3.3. Let us define the set

$$C_{>\pi}^{\infty,0}(S^1, S^1) = \{e \in C^\infty(S^1, S^1); \deg(e) = 0, \text{length}(\text{Im}(e)) > \pi\}$$

and consider the map

$$\text{pr}_2 \circ \Phi: \text{Imm}^{0,*}(S^1, S^1) \rightarrow C_{>\pi}^{\infty,0}(S^1, S^1),$$

where pr_2 denotes the second projection.

Lemma. *The map $\text{pr}_2 \circ \Phi: \text{Imm}^{0,*}(S^1, S^1) \rightarrow C_{>\pi}^{\infty,0}(S^1, S^1)$, is surjective, has contractible fibers, admits a global smooth section, and is a homotopy equivalence.*

Proof. For a map $e \in C_{>\pi}^{\infty,0}(S^1, S^1)$, there exist points $\theta_1, \theta_2, \theta_3$ such that $0 \in \text{int}([e(\theta_1), e(\theta_2), e(\theta_3)])$, where $[\cdot, \cdot, \cdot]$ denotes the convex hull of three points. Let $v_1 \in C^\infty(S^1, \mathbb{R}_{>0})$ be a map such that $\int_0^{2\pi} v_1 d\theta = 1$ and $v_1(\theta)$ is close to 0 if θ is not close to θ_1 . Then $\int_0^{2\pi} v_1.e d\theta$ is close to $e(\theta_1)$. We also define v_2 and v_3 similarly, so that

$$0 \in \text{int} \left(\left[\int_0^{2\pi} v_1.e d\theta, \int_0^{2\pi} v_2.e d\theta, \int_0^{2\pi} v_3.e d\theta \right] \right).$$

Therefore there exist positive numbers a_1, a_2, a_3 with

$$a_1 \int_0^{2\pi} v_1.e d\theta + a_2 \int_0^{2\pi} v_2.e d\theta + a_3 \int_0^{2\pi} v_3.e d\theta = 0.$$

Define c by

$$c(\theta) = \int_0^\theta (a_1 v_1(u) + a_2 v_2(u) + a_3 v_3(u)) e(u) du.$$

Then c is in $\text{Imm}^{0,*}$ and $(\text{pr}_2 \circ \Phi)(c) = e$, which means that $\text{pr}_2 \circ \Phi$ is surjective.

We next show that for any $e \in C_{>\pi}^{\infty,0}(S^1, S^1)$, the inverse image $(\text{pr}_2 \circ \Phi)^{-1}(e)$ is contractible. Namely, let $V(e) \subset C^\infty(S^1, \mathbb{R}_+)$ be given by

$$V(e) = \left\{ v \in C^\infty(S^1, \mathbb{R}_+); \int_0^{2\pi} v.e d\theta = 0 \right\},$$

an open convex subset of the linear subspace $\{v \in C^\infty(S^1, \mathbb{R}); \int_0^{2\pi} v.e d\theta = 0\} \subset C^\infty(S^1, \mathbb{R})$. Thus $V(e)$ is contractible for each e . Moreover, $V(e)$ is homeomorphic to $(\text{pr}_2 \circ \Phi)^{-1}(e)$ by the map $\text{pr}_1 \circ \Phi: (\text{pr}_2 \circ \Phi)^{-1}(e) \rightarrow V(e)$.

For fixed $\theta_1, \theta_2, \theta_3$ the construction above works for each $e \in C_{>\pi}^{\infty,0}(S^1, S^1)$ for which 0 is contained in the interior of the convex hull of $e(\theta_1), e(\theta_2), e(\theta_3)$; these e form an open set in $C_{>\pi}^{\infty,0}(S^1, S^1)$ on which we get a continuous (even smooth) section of $\text{pr}_2 \circ \Phi$. Open sets like that cover $C_{>\pi}^{\infty,0}(S^1, S^1)$. So we get smooth local sections whose domains cover the base. Since the base is open in a nuclear Fréchet space, it is smoothly paracompact (see [3], 16.10) we can use convexity of all fibers

and a smooth partition of unity on the base $C_{>\pi}^{\infty,0}(S^1, S^1)$ to construct a global smooth section s .

Finally, since all fibers are convex, there is a smooth strong fiber preserving deformation retraction of $\text{Imm}^{0,*}(S^1, S^1)$ onto the image the global section s . \square

3.4. To study the topology of $C_{>\pi}^{\infty,0}(S^1, S^1)$, we introduce the set of 2π -periodic functions

$$C^{\infty,p}(\mathbb{R}, \mathbb{R}) = \{c \in C^\infty(\mathbb{R}, \mathbb{R}); c(\theta + 2\pi) = c(\theta)\}.$$

For $c \in C^{\infty,p}(\mathbb{R}, \mathbb{R})$ let $\text{Var}(c) = \max c - \min c$ and let $\text{Ave}(c) = \frac{1}{2\pi} \int_0^{2\pi} c d\theta$. For $k \geq 0$, $C_{>k}^{\infty,p}(\mathbb{R}, \mathbb{R}) = \{c \in C^{\infty,p}(\mathbb{R}, \mathbb{R}); \text{Var}(c) > k\}$. Define a diffeomorphism $g: C_{>0}^{\infty,p}(\mathbb{R}, \mathbb{R}) \rightarrow C_{>\pi}^{\infty,p}(\mathbb{R}, \mathbb{R})$ by

$$g(c) = \frac{\text{Var}(c) + \pi}{\text{Var}(c)}(c - \text{Ave}(c)) + \text{Ave}(c).$$

The diffeomorphism g satisfies $g(c(\theta + 2n\pi)) = g(c)(\theta + 2n\pi)$, thus induces the diffeomorphism

$$\tilde{g}: C_{>0}^{\infty,0}(S^1, S^1) \rightarrow C_{>\pi}^{\infty,0}(S^1, S^1),$$

where $C_{>0}^{\infty,0}(S^1, S^1)$ denotes the set of nonconstant smooth maps of degree 0 in $C^\infty(S^1, S^1)$.

3.5. We consider now the evaluation ev_1 at $1 \in S^1$ whose fiber at $1 \in S^1$ is the smooth manifold of based smooth loops of degree 0 in S^1 , with the constant loop 1 deleted:

$$(1) \quad C^{\infty,0}((S^1, 1), (S^1, 1)) \setminus \{1\} \hookrightarrow C_{>0}^{\infty,0}(S^1, S^1) \xrightarrow{\text{ev}_1} S^1$$

Lemma. *The map $\text{ev}_1 : C_{>0}^{\infty,0}(S^1, S^1) \rightarrow S^1$ is a smooth trivial fibration with a global section and smoothly contractible fibers. Moreover, it is a homotopy equivalence.*

Proof. A smooth section $s : S^1 \rightarrow C_{>0}^{\infty,0}(S^1, S^1)$ of ev_1 is given by $s(\varphi)(\theta) = \varphi \cdot \exp(i \text{Im}(\theta))$. The fiber of ev_1 over φ is the space $C^{\infty,0}((S^1, 1), (S^1, \varphi)) \setminus \{\varphi\}$ consisting of all non-constant smooth loops of degree 0 mapping 1 to φ , which is diffeomorphic to the fiber $C^{\infty,0}((S^1, 1), (S^1, 1)) \setminus \{1\}$ via multiplication by φ .

It remains to show that the fiber $C^{\infty,0}((S^1, 1), (S^1, 1)) \setminus \{1\}$ is contractible. Via lifting to the universal cover, $C^{\infty,0}((S^1, 1), (S^1, 1))$ is diffeomorphic to the space $\{f \in C^{\infty,p}(\mathbb{R}, \mathbb{R}) : f(0) = 0\}$ of periodic functions mapping 0 to 0. Via Fourier expansion $f(t) = \sum_{n \in \mathbb{Z}} a_n \exp(int)$ this is isomorphic to the space of all rapidly decreasing complex sequences $(a_k)_{k \in \mathbb{Z}}$ with $\bar{a}_{-k} = a_k$ and $\sum_k a_k = 0$. This space is isomorphic to the space \mathfrak{s} of rapidly decreasing sequences $(b_n)_{n \geq 1}$ by $a_n = b_n$ for $n \geq 1$, $a_{-n} = \bar{b}_n$, and $a_0 = 2 \text{Re}(\sum_{n \geq 1} b_n)$.

Now we have to show that this is still contractible if we remove the constant sequence 0. Then it is homotopy equivalent to its intersection with the sphere in ℓ^2 , i.e., to the space $S := \{b \in \mathfrak{s} : \sum_{n \geq 1} b_n^2 = 1\}$. But this is contractible by a standard argument which is explained on page 513 of [3] for the space of finite sequences. Namely, consider the homotopy $A : \mathfrak{s} \times [0, 1] \rightarrow \mathfrak{s}$ through isometries which is given by $A_0 = \text{Id}$ and by

$$A_t(b_1, b_2, b_3, \dots) = (b_1, \dots, b_{n-2}, b_{n-1} \cos \theta_n(t), b_{n-1} \sin \theta_n(t), \\ b_n \cos \theta_n(t), b_n \sin \theta_n(t), b_{n+1} \cos \theta_n(t), b_{n+1} \sin \theta_n(t), \dots)$$

for $\frac{1}{n+1} \leq t \leq \frac{1}{n}$, where $\theta_n(t) = \varphi(n((n+1)t - 1)) \frac{\pi}{2}$ for a fixed smooth function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ which is 0 on $(-\infty, 0]$, grows monotonely to 1 in $[0, 1]$, and equals 1 on $[1, \infty)$. The mapping A is Lipschitz continuous for each seminorm $\|b\|_k = \sup\{|b_n|n^k : n \geq 1\}$ of \mathfrak{s} with constant 2^k , and is isometric for ℓ^2 . Then $A_{1/2}(b_1, b_2, \dots) = (b_1, 0, b_2, 0, \dots)$ is in $\mathfrak{s}_{\text{odd}}$, and on the other hand $A_1(b_1, b_2, \dots) = (0, b_1, 0, b_2, 0, \dots)$ is in $\mathfrak{s}_{\text{even}}$. This is a variant of a homotopy constructed by [6]. Now $A_t|_S$ for $0 \leq t \leq 1/2$ is a homotopy on S between the identity and $A_{1/2}(S) \subset \mathfrak{s}_{\text{odd}}$. The latter set is contractible, for example in a stereographic chart. \square

3.6. If we put together all mappings constructed above we get the following commutative diagram where we indicate isomorphism \cong , homotopy equivalence \sim , or 2-sheeted covering 2, and a free orbit $S^1.c$ for the rotation action on Imm^0 :

$$\begin{array}{ccccccc}
 & & S^1 & \xrightarrow{=} & S^1 & \xrightarrow{2} & S^1 \\
 & \nearrow \mathbb{R} & \uparrow \alpha & & \uparrow \alpha & & \uparrow \bar{\alpha} \\
 S^1.c & \xrightarrow{C} & \text{Imm}^0 & \xrightarrow{\cong} & B^{0,+} & \xrightarrow{2} & B^0 \\
 \uparrow = & & \downarrow \sim \downarrow \tilde{g}^{-1} \circ \text{pr}_2 \circ \Phi & & & & \\
 S^1 & \xleftarrow{\sim} & C_{>0}^{\infty,0} & & & &
 \end{array}$$

4. THE HOMOTOPY TYPE OF $B^0(S^1, \mathbb{R}^2)$

4.1. Proposition. *The mapping $\text{Imm}^0(S^1, \mathbb{R}^2) \rightarrow B^{0,+}(S^1, \mathbb{R}^2)$ is a (Serre) fibration.*

Proof. First we replace $\text{Imm}^0(S^1, \mathbb{R}^2)$ by the subset $\text{Imm}_a^0(S^1, \mathbb{R}^2)$ consisting of all immersions which are parametrized by scaled arc-length which is a strong deformation retract, see [4], 2.6. The normalizer of the $\text{Diff}^+(S^1)$ -action on it is just the action of S^1 which shifts the initial point. We have to show that for any compactly generated space P and a homotopy $h : [0, 1] \times P \rightarrow B^{0,+}$ whose initial value $h(0, \cdot)$ admits a continuous lift there exists a continuous lift of the whole homotopy:

$$\begin{array}{ccc}
 \{0\} \times P & \xrightarrow{H(0, \cdot)} & \text{Imm}_a^0(S^1, \mathbb{R}^2) \\
 \downarrow C & \nearrow H & \downarrow \\
 [0, 1] \times P & \xrightarrow{h} & B^{0,+}(S^1, \mathbb{R}^2)
 \end{array}$$

To get the lift H we just have to specify the initial point coherently from $H(0, p)(1)$ over $[0, 1] \ni t \mapsto h(t, p)$.

For that we need a description of the elements in $B^{0,+}(S^1, \mathbb{R}^2)$. A point C in it can be described by the following data:

For some n and $i = 1, \dots, n$, there are open sets $U_i = U_i(C) \subseteq \mathbb{R}^2$, smooth functions $f_i = f_i(C) : U_i \rightarrow \mathbb{R}$ such that $f_i^{-1}(0) =: C_i$ is a component C_i of C with $\text{grad}(f_i)$ is a unit vector field with flow lines unit speed straight lines passing orthogonally through C_i in such a way that for $x \in C_i$ the frame consisting of $\text{grad}(f_i)(x)$ and the unit tangent to C_i at x is positively oriented. The unparameterized smooth oriented 1-manifolds C_1, C_2, \dots, C_n (in that order) describe C . Note that there is a choice for the U_i and their cyclic order, but then the f_i are unique.

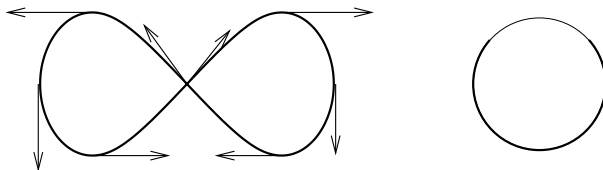
For every $p \in P$ the initial point $H(0, p)(1)$ lies in some component $h(0, p)_i$ of $h(0, p)$, and we may move it orthogonally along $\text{grad}(f_i(h(t, p)))$ to get a coherent choice of initial points. This takes care of the lift H . \square

4.2. Lemma. *The fiber $\text{Diff}^+(S^1)$ maps homotopically trivial into the fibration $\text{Imm}^0(S^1, \mathbb{R}^2) \rightarrow B^{0,+}(S^1, \mathbb{R}^2)$.*

Proof. As in the proof of 4.1 we consider the space $\text{Imm}_a^0(S^1, \mathbb{R}^2)$ of degree 0 immersions with constant speed parametrizations. Let c be the unit speed parameterized horizontal figure eight, and consider the diagram where $c_*(f) = c \circ f$:

$$\begin{array}{ccccc}
 S^1 & \xrightarrow{c_*} & \text{Imm}_a^0 & & \\
 \sim \downarrow c & & \sim \downarrow c & \searrow & \\
 \text{Diff}^+(S^1) & \xrightarrow{c_*} & \text{Imm}^0 & \longrightarrow & B^{0,+} \\
 & & \sim \downarrow \tilde{g}^{-1} \circ \text{pr}_2 \circ \Phi & & \\
 S^1 & \xleftarrow{\text{ev}_1} & C_{>0}^{\infty,0} & &
 \end{array}$$

We have to show that the mapping from the upper left S^1 to the lower left S^1 is nullhomotopic. It is essentially (suppressing \tilde{g}^{-1}) given by $\beta \mapsto \frac{c'(\beta)}{|c'(\beta)|}$. From the figure



we see that this mapping covers everything below the northern polar region twice and avoids the northern polar region, so it is nullhomotopic. \square

4.3. Corollary. *We have the following homotopy groups:*

$$\begin{array}{ll}
 \pi_1(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}, & \pi_1(B^0(S^1, \mathbb{R}^2)) = \mathbb{Z}, \\
 \pi_2(B^{0,+}(S^1, \mathbb{R}^2)) = \mathbb{Z}, & \pi_2(B^0(S^1, \mathbb{R}^2)) = \mathbb{Z}, \\
 \pi_k(B^{0,+}(S^1, \mathbb{R}^2)) = 0, & \pi_k(B^0(S^1, \mathbb{R}^2)) = 0 \quad \text{for } k > 2.
 \end{array}$$

Proof. By 4.1 we have the long exact homotopy sequence

$$\dots \rightarrow \pi_k(S^1) \xrightarrow{0} \pi_k(\text{Imm}_a^0) \rightarrow \pi_k(B^{0,+}) \rightarrow \pi_{k-1}(S^1) \rightarrow \dots$$

and by section 3 the space Imm_a^0 is homotopy equivalent to S^1 . This gives the homotopy groups of $B^{0,+}(S^1, \mathbb{R}^2)$. Since $B^{0,+}(S^1, \mathbb{R}^2) \rightarrow B^0(S^1, \mathbb{R}^2)$ is a two-sheeted covering, we can also read off the homotopy groups of $B^0(S^1, \mathbb{R}^2)$. \square

REFERENCES

- [1] W.C. Graustein. A new form of the four vertex theorem. *Mh. Math. Physics*, 43 (1936), 381–384.
- [2] W.C. Graustein. Extensions of the four vertex theorem. *Trans. AMS*, 41 (1937), 9–23.
- [3] A. Kriegl and P. W. Michor. *The Convenient Setting for Global Analysis*. AMS, Providence, 1997. Surveys and Monographs 53.

- [4] P. W. Michor and D. Mumford. Riemannian geometries on spaces of plane curves. *J. Eur. Math. Soc. (JEMS)* 8 (2006), 1-48, arXiv:math.DG/0312384.
- [5] P. W. Michor and D. Mumford. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. *Documenta Math.* 10 (2005), 217-245. arXiv:math.DG/0409303.
- [6] T. R. Ramadas. On the space of maps inducing isomorphic connections. *Ann. Inst. Fourier* 32 (1982), 263–276.
- [7] G. W. Whitehead. *Elements of homotopy theory*. GTM 61, Springer-Verlag, 1978
- [8] H. Whitney. Regular families of curves I, II. *Proc. Nat. Acad. Sci USA*, 18 (1932), 275–278, 340–342.

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